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MODELING OF NONLINEAR ANTIPLANE STRAIN OF A CYLINDRICAL BODY

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Antiplane strain of an elastic cylindrical body is studied with allowance for geometrical and physical nonlinearities and potential forces. The nonlinear boundary-value problem for two independent strains is solved. An analytical solution and the corresponding load are obtained for the Rivlin–Saunders quadratic elastic potential, which models finite elastic strains. The problem for displacements specified on the boundary is solved. The case of weak physical nonlinearity is considered.

Key words: *displacement, strains, stresses, potential, nonlinearity, boundary-value problem, analytical solution.*

To provide sufficient accuracy in studies of the strain of an elastic solid for the case where strains are finite and the material behavior does not obey Hooke's law, to it is necessary to drop the constraints of the linear theory and consider the strain with allowance for geometrical and physical nonlinearities. In the present paper, this approach is used to study the nonlinear antiplane strain of an isotropic cylindrical solid under the action of body forces in the actual-state variables.

For antiplane strain of a cylindrical solid body, the displacement is parallel to the generatrix and does not vary along the body [1, 2]. We consider this strain using the model of an incompressible nonlinear-elastic solid, which comprises equilibrium equations, Murnaghan's law, a relation between the elastic potential and the basis strain invariants, expressions for the strain components and invariants in terms of displacements, and an incompressibility condition. In the actual variables x_1, x_2, and x_3 ($x_1 = x$ and $x_2 = y$ are the transverse coordinates and $x_3 = z$ is the longitudinal coordinate), these relations are written as [3, 4]

$$F_k + \frac{\partial P_{kl}}{\partial x_l} = 0; \tag{1}$$

$$P_{kl} = -q_* \delta_{kl} + (\delta_{km} - 2E_{km}) \frac{\partial U}{\partial E_{lm}}; \tag{2}$$

$$U = U(E_1, E_2, E_3); \tag{3}$$

$$2E_{kl} = \frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l} - \frac{\partial u_m}{\partial x_k} \frac{\partial u_m}{\partial x_l}; \tag{4}$$

$$E_1 = E_{nn}, \qquad 2E_2 = E_{nn}E_{mm} - E_{nm}E_{mn}, \qquad E_3 = |E_{kl}|; \tag{5}$$

$$2E_1 - 4E_2 + 8E_3 = 0, \tag{6}$$

where q_* is the Lagrange multiplier, U is the elastic potential, E_1, E_2, and E_3 are the basis strain invariants, F_k and u_k are the components of the body force and displacement, respectively, P_{kl} and E_{kl} are the components of

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the Cauchy stresses and Almansi strains, respectively, and δ_{kl} is the Kronecker symbol (the subscripts take values 1, 2, and 3 and the summation is performed over repeated subscripts).

It is assumed that the displacement has only an axial component which is unchanged along the body and the body forces have a potential V which, like the displacement, is a function of only the transverse coordinates:

$$u_1 = 0, \qquad u_2 = 0, \qquad u_3 = w(x, y); \tag{7}$$

$$F_1 = -\frac{\partial V}{\partial x}, \qquad F_2 = -\frac{\partial V}{\partial y}, \qquad F_3 = -\frac{\partial V}{\partial z} = 0, \qquad V = V(x, y). \tag{8}$$

For displacements (7), strains (4) are related to the axial displacement by nonlinear formulas (geometrical nonlinearity) and depend on the transverse coordinates:

$$\begin{gathered} 2E_{11} = -\Big(\frac{\partial w}{\partial x}\Big)^2, \qquad 2E_{22} = -\Big(\frac{\partial w}{\partial y}\Big)^2, \qquad 2E_{33} = 0, \\ 2E_{12} = -\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \qquad 2E_{31} = \frac{\partial w}{\partial x}, \qquad 2E_{32} = \frac{\partial w}{\partial y} \qquad [E_{kl} = E_{kl}(x, y)]. \end{gathered} \tag{9}$$

Eliminating the displacement from (9), we obtain the compatibility equations for antiplane strain:

$$2E_{11} = -(2E_{31})^2, \qquad 2E_{22} = -(2E_{32})^2, \qquad 2E_{33} = 0, \qquad 2E_{12} = -2E_{31}2E_{32}; \tag{10}$$

$$\frac{\partial E_{32}}{\partial x} - \frac{\partial E_{31}}{\partial y} = 0. \tag{11}$$

The final equations (10) express the strains in terms of the independent components E_{31} and E_{32}, which are related by the differential equation (11).

The basis strain invariants (5) are expressed in terms of the displacement as

$$2E_1 = -|\nabla w|^2, \qquad 4E_2 = -|\nabla w|^2, \qquad 8E_3 = 0. \tag{12}$$

Invariants (12) are independent of the longitudinal coordinate, nonpositive, representable in terms of the linear invariant, and satisfy the incompressibility condition (6):

$$E_k = E_k(x, y), \qquad E_k \leqslant 0, \qquad E_k = E_k(E_1), \qquad 2E_1 - 4E_2 + 8E_3 = 0, \tag{13}$$

i.e., for antiplane strain, the material behaves like an incompressible material, which justifies the use of the incompressible model.

The elastic potential (3) depends only on the first invariants by virtue of the properties of invariants (13), and the tensor gradient of the potential with respect to strains is a spherical tensor:

$$\begin{gathered} U(E_1, E_2, E_3) = U(E_1), \qquad E_1 = E_{lm}\delta_{ml}, \\ \frac{\partial E_1}{\partial E_{lm}} = \delta_{ml}, \qquad \frac{\partial U}{\partial E_{lm}} = U'(E_1)\delta_{ml}. \end{gathered} \tag{14}$$

Under conditions (14), Murnaghan's law (2) is written as a quasilinear stress–strain relation (physical nonlinearity):

$$P_{kl} = -q\delta_{kl} - U'(E_1)2E_{kl}, \qquad q = q_* - U' \tag{15}$$

(q is the hydrostatic pressure). Below, it is assumed that like the displacement and force potential, the pressure depends only on the transverse coordinates: $q = q(x, y)$. Hence, it follows that the stresses are functions of these coordinates (15): $P_{kl} = P_{kl}(x, y)$.

According to the strain-compatibility relations (10), the strain components and, hence, the linear strain invariant are expressed in terms of two independent components E_{31} and E_{32}. Consequently, the stress components (15) can be written as functions of pressure and independent strains:

$$\begin{gathered} P_{11} = -q + U'(E_1)(2E_{31})^2, \qquad P_{22} = -q + U'(E_1)(2E_{32})^2, \qquad P_{33} = -q, \\ P_{12} = U'(E_1)2E_{31}2E_{32}, \qquad P_{31} = -U'(E_1)2E_{31}, \qquad P_{32} = -U'(E_1)2E_{32}; \end{gathered} \tag{16}$$

$$2E_1 = 2E_{nn} = -(2E_{31})^2 - (2E_{32})^2. \tag{17}$$

For antiplane strain, we show that the pressure can be expressed in terms of the force and elastic potentials and the independent strains are determined from the nonlinear boundary-value problem. To this end, taking into account the expressions for forces (8) and stresses (16) and bearing in mind that these quantities depend only on the transverse coordinates, we write the equilibrium equations (1) in expanded form. With allowance for the third equilibrium equation

$$\frac{\partial P_{31}}{\partial x}+\frac{\partial P_{32}}{\partial y}=-\frac{\partial U' 2E_{31}}{\partial x}-\frac{\partial U' 2E_{32}}{\partial y}=0 \tag{18}$$

we write the first two equations as

$$\begin{aligned}\frac{\partial\,(P_{11}-V)}{\partial x}+\frac{\partial P_{12}}{\partial y}&=-\frac{\partial\,(q+V)}{\partial x}+\frac{\partial 2E_{31}(U' 2E_{31})}{\partial x}+\frac{\partial 2E_{31}(U' 2E_{32})}{\partial y}\\&=-\frac{\partial\,(q+V)}{\partial x}+U'\Big(2E_{31}\,\frac{\partial 2E_{31}}{\partial x}+2E_{32}\,\frac{\partial 2E_{31}}{\partial y}\Big)=0;\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{\partial P_{12}}{\partial x}+\frac{\partial\,(P_{22}-V)}{\partial y}&=-\frac{\partial\,(q+V)}{\partial y}+\frac{\partial 2E_{32}(U' 2E_{32})}{\partial y}+\frac{\partial 2E_{32}(U' 2E_{31})}{\partial x}\\&=-\frac{\partial\,(q+V)}{\partial y}+U'\Big(2E_{32}\,\frac{\partial 2E_{31}}{\partial y}+2E_{31}\,\frac{\partial 2E_{32}}{\partial x}\Big)=0.\end{aligned} \tag{20}$$

Transforming equalities (19) and (20) with allowance for the strain-compatibility equation (11), we obtain

$$-\frac{\partial(q+V)}{\partial x}+\frac{U'}{2}\,\frac{\partial}{\partial x}\Big[(2E_{31})^2+(2E_{32})^2\Big]=0,$$

$$-\frac{\partial(q+V)}{\partial y}+\frac{U'}{2}\,\frac{\partial}{\partial y}\Big[(2E_{31})^2+(2E_{32})^2\Big]=0.$$

Taking into account the expression for the linear strain invariant (17), we write the second terms in these equations as

$$\frac{U'}{2}\,\frac{\partial}{\partial x}\Big[(2E_{31})^2+(2E_{32})^2\Big]=-U'\,\frac{\partial E_1}{\partial x}=-\frac{\partial U}{\partial x},$$

$$\frac{U'}{2}\,\frac{\partial}{\partial y}\Big[(2E_{31})^2+(2E_{32})^2\Big]=-U'\,\frac{\partial E_1}{\partial y}=-\frac{\partial U}{\partial y}$$

and finally write the equations as

$$\frac{\partial}{\partial x}(q+V+U)=0,\qquad \frac{\partial}{\partial y}(q+V+U)=0.$$

Integration of these equations yields the pressure expressed in terms of the force and elastic potentials with accuracy to an additive constant:

$$q=h-V-U,\qquad h=\text{const}. \tag{21}$$

It follows from (16) and (21) that the constant can be determined from the specified potentials and the axial component of the resulting load in the end cross section S of the cylinder

$$P_3=\int\limits_S P_{33}\,dS=-\int\limits_S q\,dS=-Sh+\int\limits_S (V+U)\,dS,\qquad h=\frac{1}{S}\Big(\int\limits_S (V+U)\,dS-P_3\Big).$$

If the resulting axial load vanishes, the constant is equal to the mean value of the potentials in the cross section of the body

$$h=\frac{1}{S}\int\limits_S (V+U)\,dS\qquad \text{for}\quad P_3=0, \tag{22}$$

and the pressure (21) is equal to the deviation of the sum of the potentials from its mean value.

The equilibrium equation (18) and the strain-compatibility equation (11) form a nonlinear system for the independent strains determined in the cross section S of the body:

$$\frac{\partial\,(U'E_{31})}{\partial x}+\frac{\partial\,(U'E_{32})}{\partial y}=0,\qquad \frac{\partial E_{32}}{\partial x}-\frac{\partial E_{31}}{\partial y}=0,$$
$$U'=U'(E_1),\qquad E_1=-2(E_{31}^2+E_{32}^2). \tag{23}$$

We write these equations in expanded form

$$H_1=(U'-4U''E_{31}^2)\,\frac{\partial E_{31}}{\partial x_1}-4U''E_{31}E_{32}\Big(\frac{\partial E_{32}}{\partial x_1}+\frac{\partial E_{31}}{\partial x_2}\Big)+(U'-4U''E_{32}^2)\,\frac{\partial E_{32}}{\partial x_2}=0,$$

$$H_2=\frac{\partial E_{32}}{\partial x_1}-\frac{\partial E_{31}}{\partial x_2}=0$$

and consider the corresponding characteristic second-order matrix whose elements and determinant are given by [5]

$$B_{kl}=\frac{\partial H_k}{\partial\,(\partial E_{3l}/\partial x_m)}\,v_m,\qquad B=\det\,(B_{kl}).$$

In this case, we obtain

$$B_{11}=(U'-4U''E_{31}^2)v_1-4U''E_{31}E_{32}v_2,\qquad B_{22}=v_1,$$

$$B_{12}=-4U''E_{31}E_{32}v_1+(U'-4U''E_{32}^2)v_2,\qquad B_{21}=-v_1,$$

$$B=B_{11}B_{22}-B_{12}B_{21}=U'(v_1^2+v_2^2)-4U''(E_{31}v_1+E_{32}v_2)^2. \tag{24}$$

From (24) it follows that if the first two derivatives of the elastic potential have different signs, the determinant is nonvanishing:

$$B<0\qquad\text{for}\quad U'<0,\quad U''\geqslant 0,$$
$$B>0\qquad\text{for}\quad U'>0,\quad U''\leqslant 0. \tag{25}$$

In these cases, the characteristic equation $B=0$ has no real roots and, hence, system (23) is of elliptic type. For this system, the boundary-value problem with specified boundary strains is well posed.

In (25), the condition $B<0$ is satisfied, in particular, for the Rivlin–Saunders quadratic potential used to model large elastic strains of rubber-like materials [2]:

$$U=aE_1^2-bE_1+c\qquad(a>0,\quad b>0,\quad c>0,\quad E_1<0),$$

$$U'=2aE_1-b<0,\qquad U''=2a>0. \tag{26}$$

(This potential generalizes the Mooney linear potential $U=-bE_1+c$ for the same materials, which corresponds to Murnaghan's linear law.)

If forces p_k are specified on the boundary L of the cylinder cross section S, the boundary values of the independent strains can be determined. Using stresses (16) and the outward lateral normal $(n_k)=(n_1,n_2,0)$, from the equalities $p_k=P_{kl}n_l$ we obtain the following nonlinear system for these strains:

$$p_1=-qn_1+4U'E_{31}(E_{31}n_1+E_{32}n_2),\qquad p_2=-qn_2+4U'E_{32}(E_{31}n_1+E_{32}n_2),$$

$$p_3=-2U'(E_{31}n_1+E_{32}n_2),\qquad q=h-V-U, \tag{27}$$

$$U=U(E_1),\qquad U'=U'(E_1),\qquad E_1=-2(E_{31}^2+E_{32}^2)\qquad\text{on}\quad L.$$

Let us write system (27) in different form. We consider the normal (n_k), tangent (t_k), and binormal (b_k) unit vectors to the contour L and linear combinations of the independent strains f_n and f_t that are uniquely related to the independent strains:

$$(n_k)=(n_1,n_2,0),\qquad (t_k)=(t_1,t_2,0)=(-n_2,n_1,0),\qquad (b_k)=(0,0,1); \tag{28}$$

$$f_n = E_{31}n_1 + E_{32}n_2, \qquad f_t = E_{31}t_1 + E_{32}t_2 = -E_{31}n_2 + E_{32}n_2; \tag{29}$$

$$E_{31} = f_n n_1 - f_t n_2, \qquad E_{32} = f_n n_2 + f_t n_1 \quad \text{on} \quad L. \tag{30}$$

By virtue of (21) and (27)–(29), the natural components of the contour load (p_n, p_t, p_b) are given by

$$\begin{gathered} p_n = p_k n_k = V + U - h + 4U' f_n^2, \quad p_t = p_k t_k = 4U' f_n f_t, \quad p_b = p_k b_k = -2U' f_n, \\ U = U(E_1), \qquad U' = U'(E_1), \qquad E_1 = -2(E_{31}^2 + E_{32}^2) = -2(f_n^2 + f_t^2) \quad \text{on} \quad L. \end{gathered} \tag{31}$$

Consequently, the boundary values of the strains E_{31} and E_{32} can be determined in terms of f_n and f_t using formulas (30) and the quantities f_n and f_t are found from the second and third relations in (31):

$$f_t = -p_t/(2p_b), \qquad p_b + 2f_n U'(E_1) = 0 \qquad [E_1 = -2f_n^2 - p_t^2/(2p_b^2)]. \tag{32}$$

The first equality in (31) [with allowance for the solution f_t, f_n of system (32)] is a constraint imposed on the lateral load to ensure antiplane strain. In formulas (32), the quantity f_t is determined only by the load and f_n by both the load and the form of the elastic potential.

For the quadratic potential (26), it follows from (32) that the derivative is given by

$$U' = 2aE_1 - b = -b - a(4f_n^2 + p_t^2/p_b^2),$$

and the second equality in (32) is the incomplete cubic equation

$$f_n^3 + sf_n + t = 0, \qquad s = \frac{ap_t^2 + bp_b^2}{4ap_b^2}, \qquad t = -\frac{p_b}{8a} \qquad \Big(T = \frac{s^3}{27} + \frac{t^2}{4}\Big). \tag{33}$$

Since $T > 0$, Eq. (33) has a single real root [6]

$$f_n = J_+ + J_-, \qquad J_\pm = \sqrt[3]{-t/2 \pm \sqrt{T}}. \tag{34}$$

In the case of weak physical nonlinearity, where the coefficient of the quadratic term in potential (26) is much smaller than the coefficient of the linear term: $k = a/b \ll 1$, expression (34) can be linearized with respect to this small parameter. To this end, we set $a = kb$ and $f_n = f_n^0 + kf_n^1$ in Eq. (33) and retain the free term and k-linear terms:

$$8kbp_b^2(f_n^0)^3 + 2bp_b^2 f_n^0 + 2kb(p_b^2 f_n^1 + p_t^2 f_n^0) - p_b^3 = 0 \qquad (k = a/b).$$

Setting the coefficients of k^0 and k^1 equal to zero, we obtain equations that give the desired approximation

$$f_n = \frac{p_b}{2b}\Big(1 - k\,\frac{p_b^4 + b^2p_t^2}{b^2p_b^2}\Big). \tag{35}$$

Thus, for the quadratic potential (26), the boundary-value problem for independent strains comprises Eqs. (23) and boundary conditions (30) whose right sides are determined by formulas (32) and (34) [for weak physical nonlinearity, relation (34) is replaced by (35)]. The relations of the problem contain no force potential, and, therefore, the potential body forces affect the pressure but have no effect on the independent strains.

The boundary-value problem for the independent stains formulated in Cartesian coordinates can be written in polar coordinates r, v, and z ($x = r\cos v$, $y = r\sin v$, and $z = z$). In these coordinate systems, the normal and strain components are related by the formulas

$$n_1 = n_r\cos v - n_v\sin v, \qquad n_2 = n_r\sin v + n_v\cos v, \qquad n_3 = n_z,$$

$$\begin{gathered} E_{11} = E_{rr}\cos^2 v + E_{vv}\sin^2 v - E_{rv}\sin 2v, \qquad E_{22} = E_{rr}\sin^2 v + E_{vv}\cos^2 v + E_{rv}\sin 2v, \\ E_{33} = E_{zz}, \qquad E_{31} = E_{zr}\cos v - E_{zv}\sin v, \qquad E_{32} = E_{zr}\sin v + E_{zv}\cos v, \\ E_{12} = (E_{rr} - E_{vv})\sin v\cos v + E_{rv}\cos 2v \qquad (E_{31}^2 + E_{32}^2 = E_{zr}^2 + E_{zv}^2). \end{gathered} \tag{36}$$

With allowance for (10) and (36), cylindrical strain components are expressed in terms of the independent components E_{zr} and E_{zv} as

$$E_{rr} = -2E_{zr}^2, \qquad E_{vv} = -2E_{zv}^2, \qquad E_{zz} = 0, \qquad E_{rv} = -2E_{zr}E_{zv}. \tag{37}$$

In cylindrical coordinates, Murnaghan's law relates the stress and strain components by formulas similar to (15). Transformation of these formulas with allowance for relations (37) yields the stress components expressed in terms of the pressure and independent strains:

$$\begin{gathered} P_{rr} = -q + U'(2E_{zr})^2, \qquad P_{vv} = -q + U'(2E_{zv})^2, \qquad P_{zz} = -q, \\ P_{rv} = U' 2E_{zr} 2E_{zv}, \qquad P_{zr} = -U' 2E_{zr}, \qquad P_{zv} = -U' 2E_{zv}, \\ U' = U'(E_1), \qquad E_1 = E_{rr} + E_{vv} + E_{zz} = -2(E_{zr}^2 + E_{zv}^2). \end{gathered} \tag{38}$$

The pressure is defined by expression (21). The independent cylindrical strains are defined by Eqs. (23) after passing to differentiation with respect to polar coordinates using the formulas

$$\frac{\partial}{\partial x} = \cos v \, \frac{\partial}{\partial r} - \frac{\sin v}{r} \, \frac{\partial}{\partial v}, \qquad \frac{\partial}{\partial y} = \sin v \, \frac{\partial}{\partial r} + \frac{\cos v}{r} \, \frac{\partial}{\partial v}$$

and replacing E_{31} and E_{32} by E_{zr} and E_{zv}, respectively, in accordance with (36). As a result, the equations for the independent cylindrical strains become

$$\begin{gathered} \frac{\partial\,(rU'E_{zr})}{\partial r} + \frac{\partial\,(U'E_{zv})}{\partial v} = 0, \qquad \frac{\partial\,(rE_{zv})}{\partial r} - \frac{\partial E_{zr}}{\partial v} = 0, \\ U' = U'(E_1), \qquad E_1 = -2(E_{zr}^2 + E_{zv}^2). \end{gathered} \tag{39}$$

It follows from (29) and (36) that the quantities f_n and f_t in cylindrical coordinates are written as

$$f_n = E_{zr} n_r + E_{zv} n_v, \qquad f_t = -E_{zr} n_r + E_{zv} n_v. \tag{40}$$

Inversion of these formulas yields the boundary strains in the form

$$E_{zr} = f_n n_r - f_t n_v, \qquad E_{zv} = f_n n_v + f_t n_r \quad \text{on} \quad L, \tag{41}$$

where f_n and f_t are determined by the contour load and the elastic potential by formulas (32) and (34) [or (32) and (35) for weak physical nonlinearity].

Relations (39) and (41) form the boundary-value problem for the independent cylindrical strains. In some cases, it admits simple analytical solutions.

Let the strains be functions of the polar radius alone: $E_{zr}(r)$ and $E_{zv}(r)$. It follows that the derivative of the elastic potential $U'(r)$ is also a function of the radius. In this case, Eqs. (39) are simplified:

$$\frac{d\,(rU'E_{zr})}{dr} = 0, \qquad \frac{d\,(rE_{zv})}{dr} = 0.$$

These equations can be integrated for any form of the potential and their solution contains two arbitrary constants

$$rE_{zv} = A, \qquad rU'E_{zr} = B, \qquad A = \text{const}, \qquad B = \text{const}, \tag{42}$$

where only E_{zr} depends on the form of the potential.

For the elastic potential (26) and its derivative

$$U' = -b(1 - 2kE_1) = -b[1 + 4k(E_{zr}^2 + E_{zv}^2)] \qquad (k = a/b) \tag{43}$$

(b and k are the elastic constants), it follows from (42) and (43) that the strain E_{zr} should be determined from the incomplete cubic equation (where B is replaced by the constant $C = B/b$)

$$E_{zr}^3 + E_{zr} d + e = 0, \qquad d = \frac{r^2 + 4kA^2}{4kr^2}, \qquad e = \frac{C}{4kr} \qquad \Big(E = \frac{d^3}{27} + \frac{e^2}{4} > 0\Big), \tag{44}$$

which has a unique real solution [6]. Thus, the independent strains are given by

$$E_{zv} = A/r, \qquad E_{zr} = I_+ + I_-, \qquad I_\pm = \sqrt[3]{-e/2 \pm \sqrt{E}}. \tag{45}$$

For weak physical nonlinearity ($k \ll 1$), the quantity E_{zr} in (45) can be determined in a k-linear approximation from the approximate equation (44) by equating the coefficients of the zeroth and first powers of the parameter to zero:

$$E_{zr} = E_{zr}^0 + kE_{zr}^1, \qquad 4kr^2(E_{zr}^0)^3 + (r^2 + 4kA^2)(E_{zr}^0 + kE_{zr}^1) + Cr = 0.$$

As a result, the strains (45) are approximately equal to

$$E_{zv} = \frac{A}{r}, \qquad E_{zr} = -\frac{C}{r}\Big(1 - 4k\,\frac{A^2 + C^2}{r^2}\Big). \tag{46}$$

We apply the solution (46) to the problem of the relative equilibrium of a hollow cylinder $r_1 \leqslant r \leqslant r_2$ of density ρ which rotates at angular velocity ω in the absence of the resulting axial end load $P_3 = 0$ under the action of the body forces (centrifugal inertial forces) $F_1 = \rho\omega^2 x$ and $F_2 = \rho\omega^2 y$ and determine the corresponding pressure, stresses, and lateral load in the approximation considered.

In this case, the elastic and force potentials are functions of the polar radius:

$$U = c + 2b\,\frac{A^2 + C^2}{r^2}\Big(1 + 2k\,\frac{A^2 - 3C^2}{r^2}\Big), \quad U' = -b\Big(1 + 4k\,\frac{A^2 + C^2}{r^2}\Big), \quad V = e - \frac{\rho\omega^2}{2}\,r^2. \tag{47}$$

Here $e = \text{const}$ and, according to (22), the constant h in (21) calculated for $P_3 = 0$, $c = 0$, and $e = 0$ (which corresponds to $U = 0$ for $E_1 = 0$ and $V = 0$ for $r = 0$) is given by

$$h = 2b(A^2 + C^2)\Big(\frac{\log\,(r_2^2/r_1^2)}{r_2^2 - r_1^2} + 2k\,\frac{A^2 - 3C^2}{r_1^2 r_2^2}\Big) - \frac{\rho\omega^2}{4}\,(r_1^2 + r_2^2).$$

In this case, the pressure (21) becomes

$$q = h + \frac{\rho\omega^2}{2}\,r^2 - 2b\,\frac{A^2 + C^2}{r^2}\Big(1 + 2k\,\frac{A^2 - 3C^2}{r^2}\Big). \tag{48}$$

In the problem considered, it follows from (48) that the cylindrical stress components (38) depend only on the polar radius:

$$\begin{aligned}
P_{rr} &= -\Big(h + \frac{\rho\omega^2}{2}\,r^2\Big) + \frac{2b}{r^2}\Big(A^2 - C^2 + 2k\,\frac{(A^2 + C^2)^2}{r^2}\Big), \qquad P_{rv} = AC\,\frac{4b}{r^2},\\
P_{vv} &= -\Big(h + \frac{\rho\omega^2}{2}\,r^2\Big) + \frac{2b}{r^2}\Big(A^2 - C^2 - 6k\,\frac{(A^2 + C^2)^2}{r^2}\Big), \qquad P_{zr} = -C\,\frac{2b}{r},\\
P_{zz} &= -\Big(h + \frac{\rho\omega^2}{2}\,r^2\Big) + \frac{2b}{r^2}\,(A^2 + C^2)\Big(1 + 2k\,\frac{A^2 - 3C^2}{r^2}\Big), \qquad P_{zv} = A\,\frac{2b}{r}\Big(1 + 4k\,\frac{A^2 + C^2}{r^2}\Big),
\end{aligned} \tag{49}$$

and, hence, they are constant on the lateral boundary of the circular tube. In addition, formulas (49) imply that the body forces affect tensile and compressive stresses and have no effect on shear stresses. In this case, the physical nonlinearity ($k \neq 0$) has an effect on stresses of both the first and second kinds.

The outward normal to the tube is directed along the radius, and, therefore, the boundary quantities (40) have the form

$$(n_r^{(1)}, n_v^{(1)}, n_z^{(1)}) = (-1, 0, 0), \quad r = r_1, \qquad (n_r^{(2)}, n_v^{(2)}, n_z^{(3)}) = (1, 0, 0), \quad r = r_2,$$

$$f_n^{(1)} = -E_{zr}^{(1)} = \frac{C}{r_1}\Big(1 - 4k\,\frac{A^2 + C^2}{r_1^2}\Big), \qquad f_t^{(1)} = -E_{zv}^{(1)} = -\frac{A}{r_1} \qquad \text{for} \quad r = r_1,$$

$$f_n^{(2)} = E_{zr}^{(2)} = -\frac{C}{r_2}\Big(1 - 4k\,\frac{A^2 + C^2}{r_2^2}\Big), \qquad f_t^{(2)} = E_{zv}^{(2)} = \frac{A}{r_2} \qquad \text{for} \quad r = r_2.$$

These strains and the derivative of the elastic potential (47) correspond to the lateral load (31)

$$\begin{aligned}
p_b^{(1)} &= \frac{2bC}{r_1}, \qquad p_t^{(1)} = \frac{4bAC}{r_1^2} \qquad \text{for} \quad r = r_1,\\
p_b^{(2)} &= -\frac{2bC}{r_2}, \qquad p_t^{(2)} = \frac{4bAC}{r_2^2} \qquad \text{for} \quad r = r_2,
\end{aligned} \tag{50}$$

which is insensitive to the nonlinearity of the elastic potential in the case considered. The load components are related by the formulas

$$p_b^{(1)}/p_b^{(2)} = -r_2/r_1, \qquad p_t^{(1)}/p_t^{(2)} = r_2^2/r_1^2,$$

which imply that the load takes an independent value on one boundary, for example, the internal boundary. Using this load, the integration constants A and C are expressed as

$$A = r_1 p_t^{(1)}/(2p_b^{(1)}), \qquad C = r_1 p_b^{(1)}/(2b). \tag{51}$$

Thus, the axisymmetric strain (46) of a hollow cylinder corresponds to the stress field (49), which depends on the polar radius, and the lateral load (50), whose axial and circumferential components are inversely related to the first and second powers of the radius. The integration constants in these formulas are defined by equalities (51).

In the case where displacements are specified on the boundary of the body, it is convenient to solve the problem for displacements. In [17], the problem was studied in Cartesian coordinates; we consider it in polar variables.

Using the expressions for the Cartesian strain components in terms of the displacement gradients (9) and the relations between the strain components in Cartesian and polar coordinates (36), we obtain

$$2E_{zr} = 2E_{31}\cos v + 2E_{32}\sin v = \frac{\partial w}{\partial x}\cos v + \frac{\partial w}{\partial y}\sin v,$$

$$2E_{zv} = -2E_{31}\sin v + 2E_{32}\cos v = -\frac{\partial w}{\partial x}\sin v + \frac{\partial w}{\partial y}\cos v.$$

Passing to differentiation with respect to polar coordinates according to the formulas

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cos v - \frac{\partial w}{\partial v}\frac{\sin v}{r}, \qquad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\sin v + \frac{\partial w}{\partial v}\frac{\cos v}{r},$$

we express the cylindrical strain components in terms of the displacement gradients:

$$E_{zr} = \frac{1}{2}\frac{\partial w}{\partial r}, \qquad E_{zv} = \frac{1}{2r}\frac{\partial w}{\partial v}. \tag{52}$$

After substitution of (52) into the strains relations (39), the second relation becomes an identity and the first relation becomes the desired equation for the displacements in polar coordinates:

$$\begin{gathered} \frac{\partial}{\partial r}\Big(rU'\frac{\partial w}{\partial r}\Big) + \frac{\partial}{\partial v}\Big(\frac{1}{r}U'\frac{\partial w}{\partial v}\Big) = 0, \\ U' = U'(E_1), \qquad E_1 = -\frac{1}{2}\Big[\Big(\frac{\partial w}{\partial r}\Big)^2 + \frac{1}{r^2}\Big(\frac{\partial w}{\partial v}\Big)^2\Big]. \end{gathered} \tag{53}$$

Let us obtain an axisymmetric solution of this equation. In this case, the displacement depends only on the polar radius. Hence,

$$w = w(t), \qquad t = r^2, \qquad \frac{\partial w}{\partial r} = 2r\frac{dw}{dt}, \qquad \frac{\partial w}{\partial v} = 0,$$

$$E_1 = -2t\Big(\frac{dw}{dt}\Big)^2, \qquad U' = U'(E_1) = U'(t), \qquad \frac{\partial U'}{\partial v} = 0$$

and Eq. (53) becomes

$$\frac{d}{dt}\Big(tU'\frac{dw}{dt}\Big) = 0.$$

After integration, we obtain the integral

$$tU'\frac{dw}{dt} = B, \qquad B = \text{const}. \tag{54}$$

Use of the quadratic elastic potential (26) and replacement of the constant B by C

$$U' = -b(1 - 2kE_1) = -b\Big[1 + 4kt\Big(\frac{dw}{dt}\Big)^2\Big], \qquad k = \frac{a}{b}, \qquad B = -bC = \text{const}$$

reduces integral (54) to the incomplete cubic equation for w_t

$$\Big(\frac{dw}{dt}\Big)^3 + u\frac{dw}{dt} + s = 0, \qquad u = \frac{1}{4kt}, \qquad s = -\frac{C}{4kt^2} \qquad \Big(S = \frac{u^3}{27} + \frac{s^2}{4} > 0\Big), \tag{55}$$

which has the unique real solution [6]

$$\frac{dw}{dt} = N_+(t,C) + N_-(t,C), \qquad N_\pm(t,C) = \sqrt[3]{-s/2 \pm \sqrt{S}}.$$

Integration of this relation yields the displacement

$$w = \int (N_+(t,C) + N_-(t,C))\,dt + D, \qquad C = \text{const}, \quad D = \text{const}. \tag{56}$$

The constants appearing in (56) are determined from the boundary displacements specified on the lateral surfaces of the body. In the particular case of weakly nonlinear elastic potential ($k \ll 1$), the derivative of the displacement can be obtained from the approximate equation (55)

$$w_t = w_t^0 + kw_t^1, \qquad 4kt^2(w_t^0)^3 + t(w_t^0 + kw_t^1) - C = 0$$

in the approximate form

$$\frac{dw}{dt} = \frac{C}{t}\Big(1 - 4k\,\frac{C^2}{t}\Big).$$

Integrating this equation, we obtain the displacement

$$w = D + C\ln t + \frac{4kC^3}{t}, \qquad C = \text{const}, \quad D = \text{const}. \tag{57}$$

We apply solution (57) to the problem of deformation of a circular cylindrical tube $t_1 \leqslant t \leqslant t_2$ on whose lateral surfaces constant displacements are specified:

$$w = w_1 \quad \text{for} \quad t = t_1, \qquad w = w_2 \quad \text{for} \quad t = t_2. \tag{58}$$

With allowance for (57), conditions (58) reduce to the following equations for the constants C and D:

$$w_1 = D + C\ln t_1 + 4kC^3/t_1, \qquad w_2 = D + C\ln t_2 + 4kC^3/t_2.$$

Summation and subtraction of these equalities yields the following relations, the first of which defines D in terms of C and the second is a cubic equation for C:

$$D = \frac{w_1 + w_2}{2} - \frac{C}{2}\ln(t_1t_2) - 2kC^3\,\frac{t_1 + t_2}{t_1t_2},$$

$$4k\,\frac{t_2 - t_1}{t_1t_2}\,C^3 - C\ln\frac{t_2}{t_1} + w_2 - w_1 = 0.$$

In the k-linear approximation, the constants have the values

$$C = \frac{w_2 - w_1}{\ln(t_2/t_1)}\Big(1 + 4k\,\frac{t_2 - t_1}{t_1t_2}\,\frac{(w_2 - w_1)^2}{\ln^3(t_2/t_1)}\Big), \qquad t = r^2,$$

$$D = \frac{w_1 + w_2}{2} - \frac{w_2 - w_1}{2}\,\frac{\ln(t_1t_2)}{\ln(t_2/t_1)} - 2k\,\frac{t_1 + t_2}{t_1t_2}\,\frac{(w_2 - w_1)^3}{\ln^3(t_2/t_1)}\Big(1 + \frac{t_2 - t_1}{t_2 + t_1}\,\frac{\ln(t_1t_2)}{\ln(t_2/t_1)}\Big).$$

If the displacement depends on both polar coordinates, the elastic potential is quadratic, and $k \ll 1$, then the approximation linear in the small parameter $w = w^0 + kw^1$ can be obtained from the approximate equation (53)

$$\frac{\partial}{\partial r}\Big[r\,\frac{\partial w^0}{\partial r} + kr\Big(\frac{\partial w^1}{\partial r} - 2E_1^0\,\frac{\partial w^0}{\partial r}\Big)\Big] + \frac{\partial}{\partial v}\Big[\frac{1}{r}\,\frac{\partial w^0}{\partial v} + \frac{k}{r}\Big(\frac{\partial w^1}{\partial v} - 2E_1^0\,\frac{\partial w^0}{\partial v}\Big)\Big] = 0,$$

$$2E_1^0 = -\Big(\frac{\partial w^0}{\partial r}\Big)^2 - \Big(\frac{1}{r}\,\frac{\partial w^0}{\partial v}\Big)^2.$$

Equating the coefficients of k^0 and k^1 to zero, after some simplifications we obtain the following equations for the displacement components

$$r\,\frac{\partial}{\partial r}\Big(r\,\frac{\partial w^0}{\partial r}\Big) + \frac{\partial^2 w^0}{\partial v^2} = 0; \tag{59}$$

$$r\,\frac{\partial}{\partial r}\Big(r\,\frac{\partial w^1}{\partial r}\Big) + \frac{\partial^2 w^1}{\partial v^2} = r^2\,\frac{\partial w^0}{\partial r}\,\frac{\partial 2E_1^0}{\partial r} + \frac{\partial w^0}{\partial v}\,\frac{\partial 2E_1^0}{\partial v}, \tag{60}$$

which are homogeneous and inhomogeneous harmonic equations, respectively. Equation (59) has the solution

$$w^0 = r \sin (v + f), \qquad f = \text{const},$$

which corresponds to $2E_1^0 = -1$. As a result, Eq. (60) becomes a homogeneous equation, which, in particular, has the solution

$$w^1 = gr \sin (v + f), \qquad g = \text{const}, \quad f = \text{const}.$$

Thus, the approximate solution of Eq. (53) containing two parameters has the form

$$w = w^0 + kw^1 = (1 + kg) r \sin (v + f) \qquad (g = \text{const}, \quad f = \text{const}). \tag{61}$$

The first term on the right side in (61) (which does not contain the parameter k) corresponds to the contribution of the linear elastic potential to the displacement and the second term accounts for the contribution of physical nonlinearity; for $kg \approx 1$, this contribution is comparable to that of the linear potential.

In the problem of a cylindrical tube $r_1 \leqslant r \leqslant r_2$, solution (61) corresponds to boundary displacements that vary according to the sinusoidal law

$$w_1 = (1 + kg) r_1 \sin (v + f), \qquad r = r_1, \qquad w_2 = (1 + kg) r_2 \sin (v + f), \qquad r = r_2.$$

The boundary displacements are proportional to the radii: $w_1/w_2 = r_1/r_2$; therefore, it suffices to examine them on one of the boundaries, for example, the internal boundary. On this boundary, the constant g determines the displacement amplitude w_1^* and the constant f determines the polar angle v_* of its maximum: $r_1(1 + kg) = w_1^*$ and $v_* + f = \pi/2$.

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